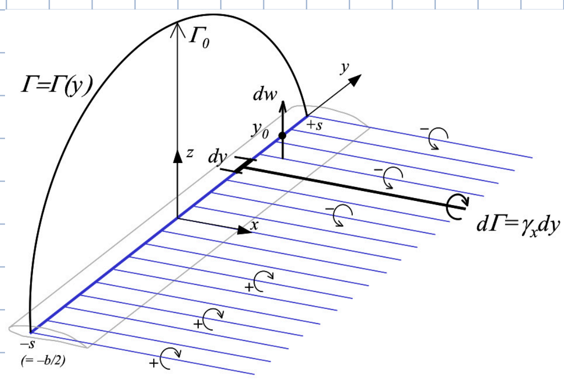
Lifting Line theory is used to obtain values for lift & drag on a 3D wing.

- Following from horseshoe vortices model
  ↳ superimposing an infinite of horseshoe vortices with strength $d\Gamma$



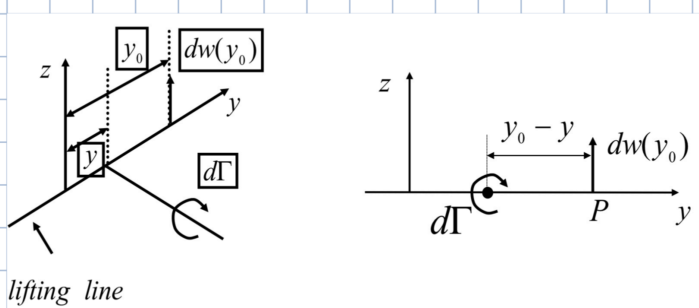
→ models a continuous bound circulation distribution $\Gamma(y)$ on lifting line and a continuous vortex shed from the wing trailing edge $\gamma_x(y)$

First we need to obtain expression for downwash velocity induced by shed wake on the bound vortex

→ The shed vortex strength = change in bound vortex strength

$$d\Gamma_{wake} = d\Gamma_{bound} = \left(\frac{d\Gamma_{bound}}{dy}\right) dy = \gamma_x(y)\, dy$$

- We now need to find the downwash velocity induced by wake element $d\Gamma$ at an arbitrary point on the bound vortex, e.g y_0



→ using Bio-Savart law for a point level with a semi-infinite vortex

$$|V| = \frac{\Gamma}{4\pi h}$$

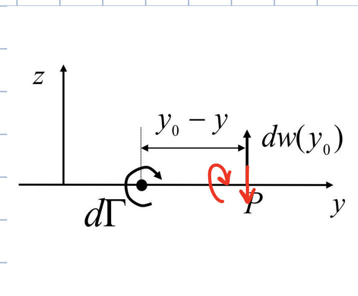
gives magnitude not direction

- We now find velocity with direction

→ velocity induced at y_0 by a filament y with +ve circulation has magnitude:

$$|V| = \frac{d\Gamma}{4\pi(y_0 - y)}$$

+ve circulation means it will produce downwash at point y_0



→ so velocity in z direction, dw, (defined +ve ↑) is:

$$dw = -\frac{d\Gamma}{4\pi(y_0 - y)} = -\frac{(d\Gamma/dy)\,dy}{4\pi(y_0 - y)}$$

↳ integrating from tip to tip:

$$\boxed{w(y_0) = -\frac{1}{4\pi}\int_{-s}^{+s} \frac{(d\Gamma/dy)\,dy}{(y_0 - y)}}$$

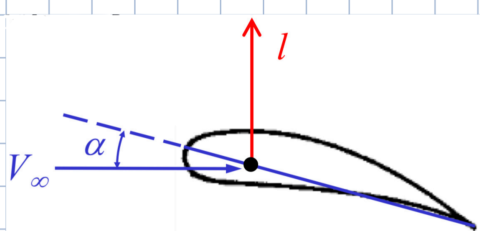
- We need to determine circulation

↳ using some 3D equivalent of Kutta condition

→ Our method involves treating each chordwise wing section as 2D aerofoil.

↳ limited to straight wings & high aspect ratios

→ Starting with basic 2D aerofoil:



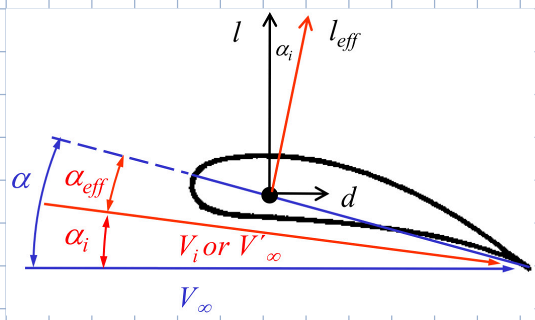
$l = \rho_\infty V_\infty \Gamma$, $d = 0$ (d'Alembert paradox)

$C_l = a_0(\alpha - \alpha_{L=0})$

↳ a_0 = lift-curve slope ≈ 2π

→ for a 3D wing, effective incidence is reduced by α_i due to downwash velocity induced by shed vortices.

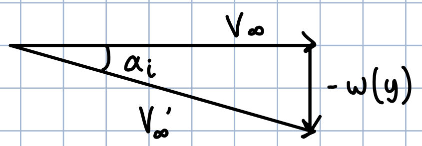
↳ lift vector rotated clockwise by α_i → L_{eff}



lift & drag then become:

$$\boxed{\begin{aligned} l &= l_{eff}\cos\alpha_i \approx l_{eff} \\ d &= l_{eff}\sin\alpha_i \approx l_{eff}\,\alpha_i \end{aligned}}$$

& $\alpha_{eff} = \alpha - \alpha_i$



$\alpha_i \approx \frac{-w}{V_\infty}$, $V_\infty' \approx V_\infty$ (small angles)

→ subbing $w(y)$ into equation:

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-s}^{+s}\frac{(d\Gamma/dy)\,dy}{(y_0-y)}$$

- using 2D flow approximation for thin wings, local coefficient is:

$$C_l(y_0) = a_0(y_0)\left[\alpha_{eff}(y_0) - \alpha_{l=0}(y_0)\right] = a_0\left[\alpha - \alpha_i(y_0) - \alpha_{l=0}(y_0)\right] \qquad (1)$$

- from Kutta-Joukowsky another expression for local lift coefficient is:

$$C_l(y_0) = \frac{\cancel{\rho_\infty V_\infty}\,\Gamma(y_0)}{\frac{1}{2}\cancel{\rho_\infty} V_\infty^{\cancel{2}}\,c(y_0)} = \frac{2\,\Gamma(y_0)}{V_\infty\,c(y_0)} \qquad (2)$$

→ equating to expressions & subbing α_i in, gives us fundamental lifting line equation:

$$\boxed{\alpha(y_0) = \frac{2\Gamma(y_0)}{a_0(y_0)\,V_\infty\,c(y_0)} + \alpha_{l=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-s}^{+s}\frac{(d\Gamma/dy)\,dy}{(y_0-y)}}$$

→ for known aerofoil & incidence, only unknown to be solved for is circulation distribution Γ.

- Assuming we can solve integro-differential equation for $\Gamma(y_0)$

→ spanwise lift per unit span is

$$l(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

therefore total lift is :

$$\boxed{L = \rho_\infty V_\infty \int_{-s}^{+s} \Gamma(y)\, dy \quad \text{and} \quad C_L = \frac{2}{V_\infty S}\int_{-s}^{+s} \Gamma(y)\, dy}$$

Corresponding drag per unit span is

$$d(y_o) \approx l(y_o)\,\alpha_i(y_o)$$

Therefore total induced drag is :

$$\boxed{D_i = \rho V_\infty \int_{-s}^{+s} \Gamma(y)\alpha_i(y)\, dy \quad \& \quad C_{D_i} = \frac{2}{V_\infty S}\int_{-s}^{+s} \Gamma(y)\alpha_i(y)\, dy}$$

Elliptical Circulation Distribution :

- Important special case where

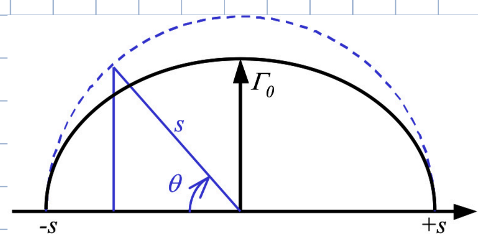
$$\boxed{\Gamma(y) = \Gamma_o \sqrt{1 - \left(\frac{y}{s}\right)^2}}$$

constant

Plugging into $w(y_o) = \frac{\Gamma_o}{4\pi s^2}\int_{-s}^{+s} \frac{y}{\sqrt{1 - y^2/s^2}\,(y_o - y)}\, dy$

using substitution :

$$y = -s\cos\theta \quad , \quad dy = s\sin\theta\, d\theta$$



$$\rightarrow \quad w(\theta_o) = \frac{-\Gamma_o}{4\pi s}\int_0^{\pi} \frac{\cos\theta}{\cos\theta - \cos\theta_o}\, d\theta$$

$$\boxed{w(\theta_o) = \frac{-\Gamma_o}{4s}}$$

$\rightarrow$ if downwash constant, so is induced incidence :

$$\alpha_i(\theta_o) = -\frac{w(\theta_o)}{V_\infty} = \boxed{\frac{\Gamma_o}{4V_\infty s}} \; *$$

Total Lift for Elliptical Distribution :

$$\boxed{C_L} = \frac{s\Gamma_o\pi}{V_\infty S} = \frac{\Gamma^*}{4V_\infty s}\frac{4s^2\pi}{S} = \alpha_i \pi \frac{(2s)^2}{S} = \alpha_i \pi \frac{b^2}{S} = \boxed{\alpha_i \pi AR}$$

$\therefore \quad \boxed{\alpha_i = \frac{1}{\pi AR} C_L}$

2D $AR = \infty \quad \therefore \alpha_i \to 0$ ✓

Total Drag for Elliptical Distribution :

$$\boxed{C_{D_i} = \frac{C_L^2}{\pi AR}}$$

'Lift dependent' or 'induced' drag $\uparrow \propto C_L^2$

↳ $\uparrow$ at $\uparrow$ lift conditions eg. take off / landing

- Elliptical circulation / lift distribution has min. induced drag.

If you have elliptical distribution on untwisted wing :

$$\alpha_i(\theta_0) = \frac{\Gamma_0}{4V_\infty s} \longrightarrow \boxed{\alpha_{eff} = \alpha - \frac{\Gamma_0}{4V_\infty s}}$$

→ local lift coefficient also constant along span

→ combination of elliptical lift distribution & constant C_l means chord distribution must be elliptical.

↳ proof in video (27:00)

- for this special case :

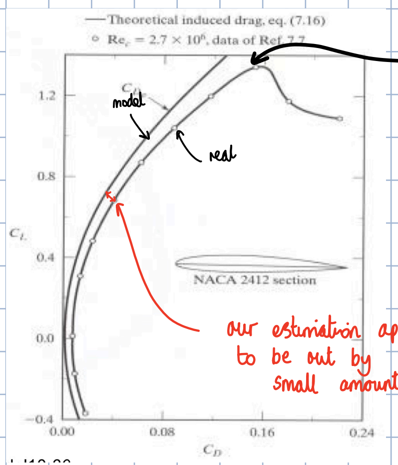
$$C_L = a_0(\alpha - \alpha_i - \alpha_{L=0}) = a_0\left(\alpha - \frac{C_L}{\pi AR} - \alpha_{L=0}\right)$$

$$\boxed{C_L = \left\{\frac{a_0}{1 + a_0/\pi AR}\right\}(\alpha - \alpha_{L=0}) = a(\alpha - \alpha_{L=0})}$$

lift-curve slope is

$$\boxed{a = \left\{\frac{a_0}{1 + a_0/\pi AR}\right\}}$$

Does theory match experimental ?



our estimation appear to be out by small amount

seperated flow here which creates large discrepancy → we use inviscid models so aren't hoping to model this

$$\boxed{C_D = C_{D_0} + KC_L^2 \quad = \quad C_{D_0} + \frac{C_L^2}{\pi AR}}$$

offset of real due to additional parasitic drag → viscous effects etc.

we are modelling this term

In the general case, circulation distribution can be modelled using a Fourier sine series :

$$\boxed{\Gamma(\theta) = 4sV_\infty \sum_{n=1}^{N} A_n \sin n\theta} \qquad \text{where } y = s\cos\theta$$

→ for elliptic case, only the 1st term A_1 is non-zero

$$\boxed{\Gamma(\theta)_{\text{elliptic}} = \Gamma_0 \sin\theta} \qquad \text{where } \boxed{A_1 = \frac{\Gamma_0}{4sV_\infty} = \alpha_i}$$

The corresponding induced incidence distribution in general is

$$\boxed{\alpha_i(\theta) = \frac{-w(\theta_0)}{V_\infty} = \sum_{n=1}^{N} nA_n \frac{\sin n\theta_0}{\sin\theta_0}}$$

→ lift depends only on A_1 : $\boxed{C_L = A_1 \pi AR}$

→ induced drag becomes : $\boxed{C_{D_i} = \frac{C_L^2}{\pi AR}[1+\delta]}$ $\delta = 0$ for elliptic

$$= \boxed{\frac{C_L^2}{e\pi AR}}$$

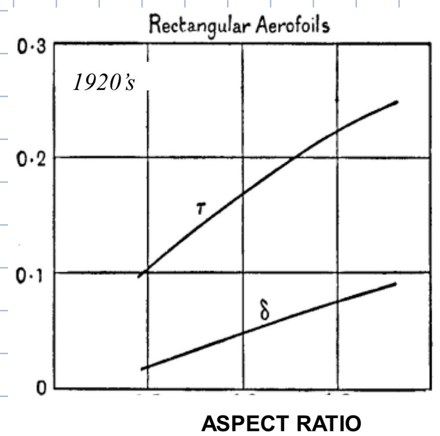
where $\boxed{e = \frac{1}{1+\delta}}$ 'span efficiency factor' $e = 1$ for elliptic

→ for untwisted wing, $C_L = a(\alpha - \alpha_{L=0})$

where $\boxed{a = \dfrac{a_0}{1 + \dfrac{a_0}{\pi AR}(1+\tau)}}$

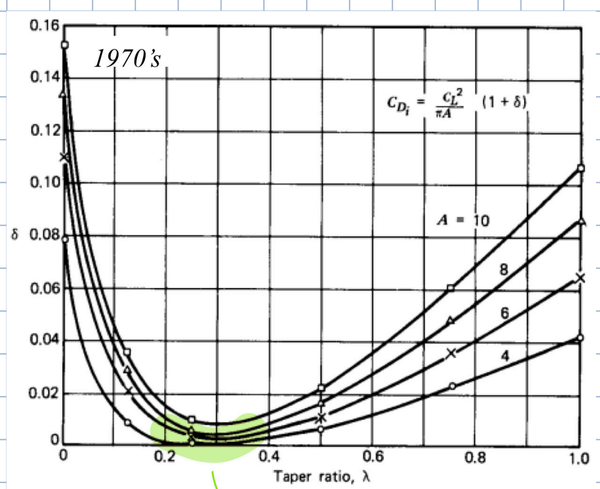
a_0 = 2D lift curve slope

Variation of τ & δ with Aspect Ratio :



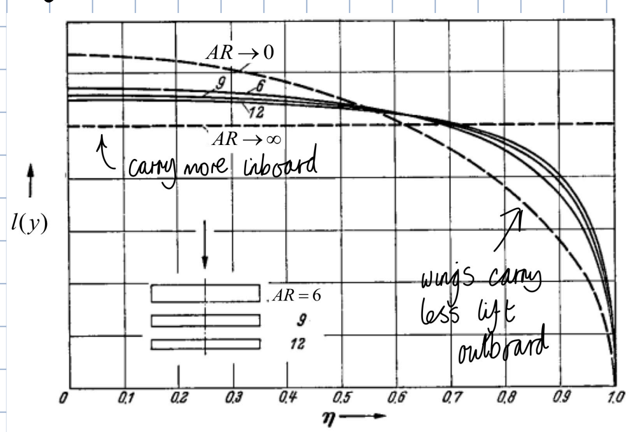
$\dfrac{\text{tip chord}}{\text{root chord}}$

Variation of δ with Taper Ratio

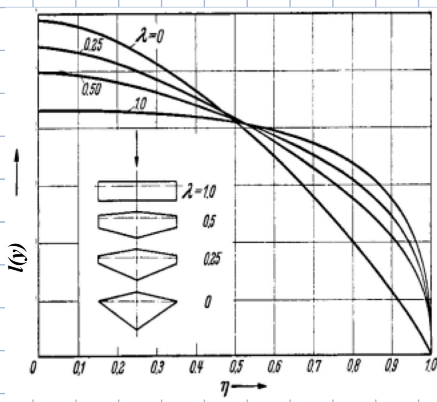


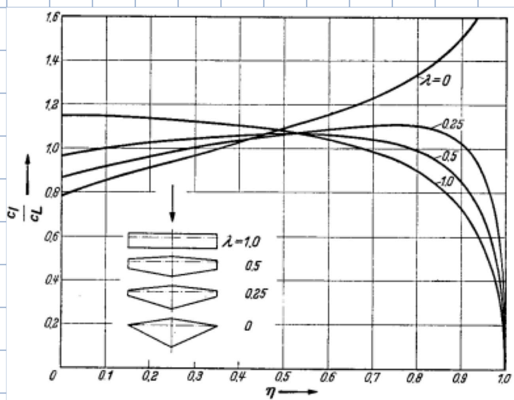
minimal induced drag

Effect of AR on Lift Distribution :



Effect of λ on Lift Distribution

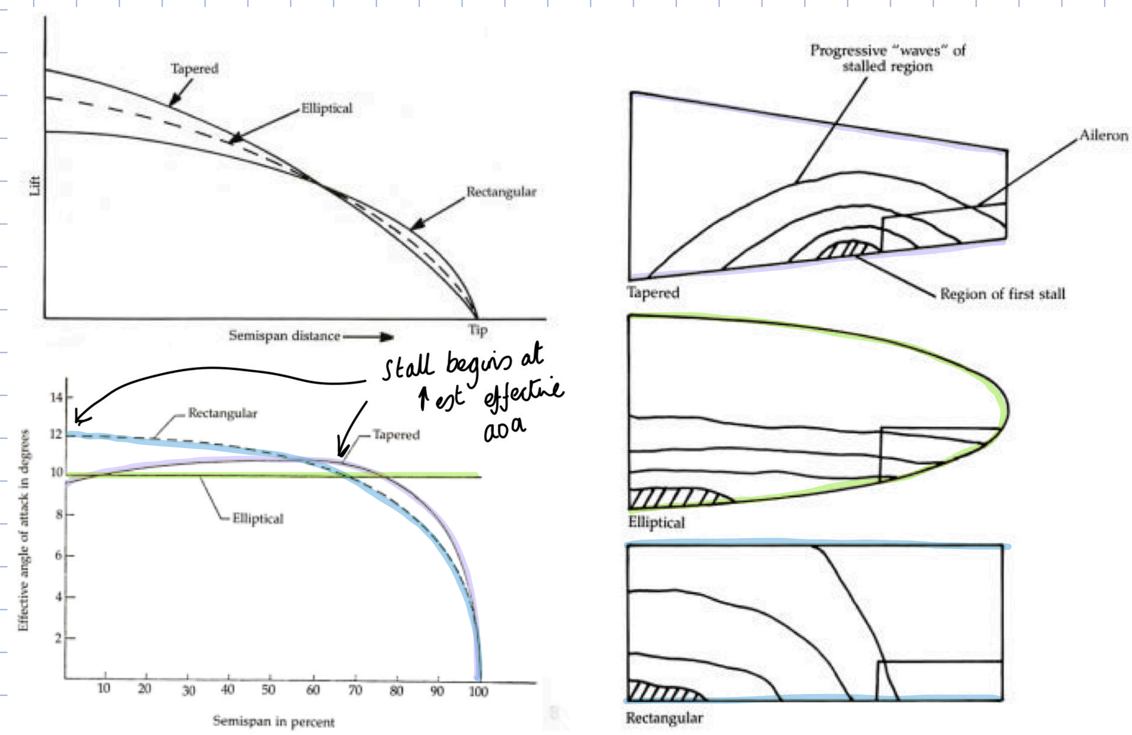




Wing Planforms:

- Untwisted wings with same aerofoil section, but different planforms → rectangular, elliptical, tapered

→ will have different chord & downwash variation along the span :



Spanwise Lift :

Effective AoA :

Elliptical : constant across span

Rectangular : highest at root

Tapered : for wing above, highest value at ~ $\frac{2}{3}$ of semi-span

Stall Behaviour :

Elliptical : stall fairly even across span

Rectangular : stall at root first

Tapered : stalls outboard position first

→ inboard stall first is desirable as creates less rolling moment.

→ outboard stall impairs ailerons

→ can use geometric twist to have lower AoA outboard

↳ or aerodynamic twist to use aerofoil sections with lower stall angle inboard.

Induced Drag Summary :

- From lifting line, we can see for given lift, induced drag lowest for elliptical planform

  ↳ elliptic wings harder to produce → expensive

- Can use taper ratios to achieve induced drag similar to elliptical

Planform Selection : based on compromise

Elliptical

✓ best induced drag

✗ most expensive.

Rectangular

✓ cheapest and has favourable stall pattern

✗ outboard section is heavier than necessary

↳ the lift load the structure must take decreases towards tip

Tapered wings

✓ can use taper ratios to achieve induced drag similar to elliptical

✗ stall pattern not favourable

↳ can be overcome with twist

↳ but this can increase parasitic drag (pressure & skin friction)

Winglets :

- Aims to reduce induced drag by breaking up tip vortices
- Other benefits :
  → during takeoff, wingtip prevented from stalling first
  → shorter take off
  → improved aileron response
  → increased stability
- They are hard to design

→ Induced drag make reduce at cost of other forms of drag
→ Can cause vibrations in main wing (buffeting)
→ can reduce yaw control